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# Positive solutions of superlinear semipositone singular Dirichlet boundary value problems<sup>☆</sup>

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## Abstract

In this paper, we study a class of superlinear semipositone singular second order Dirichlet boundary value problem. A sufficient condition for the existence of positive solution is obtained under the more simple assumptions.

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**Keywords:** Singular boundary value problem; Semipositone; Positive solutions; Superlinear

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## 1. Introduction

This paper is motivated by the boundary value problem (BVP)

$$\begin{cases} u'' + \frac{u^{3/2}}{20t(1-t)} - \frac{1}{\sqrt{t}} = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \quad (1.1)$$

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which arises naturally in chemical reactor theory [7]. Problems of this type that the nonlinearity in (1.1) may change sign are referred to as semipositone problems in the literature. In applications one is interested in showing the existence of positive solutions for this type of problems.

Motivated by the problem this paper presents existence results for

$$\begin{cases} u'' + f(t, u) + q(t) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \quad (1.2)$$

where  $f: C(0, 1) \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous,  $q(t): (0, 1) \rightarrow (-\infty, +\infty)$  is Lebesgue integrable.  $f$  may be singular at  $t = 0, 1$  and  $q$  can have finitely many singularities. Problems of form (1.2) have been discussed extensively in the literature (see [1–6] and references therein) for the case where  $q(t) \equiv 0$  (i.e., positone problem). In particular, Zhang [5] and Ma [6] considered the special case of positone BVP (1.2) when  $f(t, u) = a(t)h(u)$ ,  $q(t) \equiv 0$ . But only a handful of papers [8–10] have appeared where the nonlinearity term is allowed to change sign, moreover most of them treated with semipositone problems of the form  $f(t, u) + M \geq 0$  for some  $M > 0$ . It is value to point out that  $f$  may tend to negative infinity in this paper.

We shall organize this paper as follows. We first approximate the singular semipositone problem to the singular positone problem by a substitution. Then using the fixed point index and the Arzela–Ascoli theorem, we will complete the proof. Finally, we give an application of the existence of a positive solution for BVP (1.1) to state the rationality of our theorem.

For the convenience, we make the following assumptions:

(H<sub>1</sub>) For any  $t \in (0, 1)$ ,  $f(t, 1) > 0$ , there exist constants  $\lambda_1 \geq \lambda_2 > 1$  such that, for any  $t \in (0, 1)$ ,  $u \in [0, +\infty)$ ,

$$c^{\lambda_1} f(t, u) \leq f(t, cu) \leq c^{\lambda_2} f(t, u), \quad \forall 0 \leq c \leq 1. \quad (1.3)$$

(H<sub>2</sub>)  $\int_0^1 q_-(t) dt = r > 0$  and

$$\frac{r}{(r+1)^{\lambda_1} + 1} > 2 \int_0^1 t(1-t)[f(t, 1) + q_+(t)] dt, \quad (1.4)$$

where  $q_+(t) = \max\{q(t), 0\}$ ,  $q_-(t) = \max\{-q(t), 0\}$ .

**Remark 1.1.** It is clear that the following condition is a special case of the condition on  $q$  which implies that  $q$  can have finitely many singularities:

For given points  $t_1, t_2, \dots, t_m$ ,  $q: (0, 1) \setminus \{t_i: i = 1, \dots, m\} \rightarrow (-\infty, +\infty)$  is continuous and integrable. Furthermore,  $\int_0^1 q_-(t) dt > 0$ .

**Remark 1.2.** For any  $c \geq 1$ ,  $(t, u) \in (0, 1) \times [0, +\infty)$ , we have

$$c^{\lambda_2} f(t, u) \leq f(t, cu) \leq c^{\lambda_1} f(t, u). \quad (1.5)$$

In fact, for any  $c \geq 1$ ,  $(t, u) \in (0, 1) \times [0, +\infty)$ , by (1.3) we obtain

$$f(t, u) = f\left(t, \frac{1}{c} \cdot cu\right) \leq \left(\frac{1}{c}\right)^{\lambda_2} f(t, cu),$$

so  $c^{\lambda_2} f(t, u) \leq f(t, cu)$ . At the same time, we have  $f(t, cu) \leq c^{\lambda_1} f(t, u)$ . Thus, when  $c \geq 1$ , we have

$$c^{\lambda_2} f(t, u) \leq f(t, cu) \leq c^{\lambda_1} f(t, u).$$

If  $u_0 \in C[0, 1] \cap C^2(0, 1)$  satisfies (1.2) and  $u_0(t) > 0$ ,  $0 < t < 1$ , then we call that  $u_0(t)$  is a  $C[0, 1] \cap C^2(0, 1)$  positive solution of BVP (1.2).

We state our main result as follows:

**Theorem 1.1.** *Suppose that  $(H_1)$ ,  $(H_2)$  hold. Then BVP (1.2) has at least one  $C[0, 1] \cap C^2(0, 1)$  positive solution  $u_0(t)$ , and exists a constant  $k > 0$  such that  $u_0(t) \geq kt(1 - t)$ ,  $t \in [0, 1]$ .*

## 2. Preliminaries and lemmas

In order to overcome difficulty that the singularity and semipositone problem bring, we will use the method of the fixed point index and combining method of varying in translation to show our main result. The following lemmas play an important role to prove our result.

**Lemma 2.1** [11]. *Let  $X$  be a real Banach space,  $\Omega$  be a bounded open subset of  $X$  with  $\theta \in \Omega$  and  $A: \overline{\Omega} \cap P \rightarrow P$  is a completely continuous operator, where  $P$  is a cone in  $X$ .*

(i) *Suppose that*

$$Au \neq \lambda u, \quad \forall u \in \partial\Omega \cap P, \lambda \geq 1.$$

$$\text{Then } i(A, \Omega \cap P, P) = 1.$$

(ii) *Suppose that*

$$Au \not\leq u, \quad \forall u \in \partial\Omega \cap P.$$

$$\text{Then } i(A, \Omega \cap P, P) = 0.$$

**Lemma 2.2.** *If  $f(t, u)$  satisfies  $(H_1)$ , then for any  $t \in (0, 1)$ ,  $f(t, u)$  is increasing on  $u \in [0, +\infty)$ , and for any  $[\alpha, \beta] \subset (0, 1)$ ,*

$$\lim_{u \rightarrow +\infty} \min_{t \in [\alpha, \beta]} \frac{f(t, u)}{u} = +\infty.$$

**Proof.** For any  $t \in (0, 1)$ ,  $x, y \in [0, +\infty)$ , without loss of the generality, let  $0 \leq x \leq y$ . If  $y = 0$ , obviously  $f(t, x) \leq f(t, y)$  holds. If  $y \neq 0$ , let  $c_0 = x/y$ , then  $0 \leq c_0 \leq 1$ . It follows from (1.3) that

$$f(t, x) = f(t, c_0 y) \leq c_0^{\lambda_2} f(t, y) \leq f(t, y).$$

Thus  $f(t, u)$  is increasing on  $u$  in  $[0, \infty)$ .

On the other hand, choose  $u > 1$ . It follows from (1.5) that  $f(t, u) \geq u^{\lambda_2} f(t, 1)$ . Then

$$\frac{f(t, u)}{u} \geq u^{\lambda_2-1} f(t, 1), \quad \forall t \in (0, 1).$$

By  $(H_1)$ , for any  $[\alpha, \beta] \subset (0, 1)$ ,  $\forall t \in [\alpha, \beta]$ , we have

$$\min_{t \in [\alpha, \beta]} \frac{f(t, u)}{u} \geq \min_{t \in [\alpha, \beta]} u^{\lambda_2-1} f(t, 1) > 0.$$

Therefore

$$\lim_{u \rightarrow +\infty} \min_{t \in [\alpha, \beta]} \frac{f(t, u)}{u} = +\infty. \quad \square$$

Let  $X = C[0, 1]$  be a real Banach space equipped with the norm  $\|u\| = \max_{t \in (0, 1)} |u(t)|$  for  $u \in C[0, 1]$ . We let  $P = \{u \in C[0, 1]: u(t) \geq 0\}$  and  $Q = \{u \in P: u(t) \geq \|u\|t(1-t)\}$ . Clearly  $P, Q$  is a cone of  $C[0, 1]$ , and  $Q \subset P$ .

Now, we introduce the Green's function

$$G(t, s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$$

for the following BVP:

$$\begin{cases} u'' = 0, & t \in (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

Define the function, for  $y \in X$ ,

$$[y(t)]^* = \begin{cases} y(t), & y(t) \geq 0, \\ 0, & y(t) < 0. \end{cases}$$

Let  $x(t) = \int_0^1 G(t, s)q_-(s)ds$ ,  $0 \leq t \leq 1$ . By  $(H_2)$  we have

$$x(t) = \int_0^1 G(t, s)q_-(s)ds \leq \int_0^1 G(s, s)q_-(s)ds \leq \frac{1}{4} \int_0^1 q_-(s)ds < +\infty.$$

Since  $G(t, s) \geq 0$ , we have  $x \in P$ . By direct computation, we know  $x(0) = x(1) = 0$ ,  $x''(t) = -q_-(t)$ .

For any fixed  $u \in P$ , choose  $0 < a < 1$  such that  $a\|u\| < 1$ , then  $a[u(t) - x(t)]^* \leq au(t) \leq a\|u\| < 1$ , so by (1.5), (1.3) and Lemma 2.2, we have

$$f(t, [u(t) - x(t)]^*) \leq \left(\frac{1}{a}\right)^{\lambda_1} f(t, a[u(t) - x(t)]^*) \leq a^{\lambda_2-\lambda_1} \|u\|^{\lambda_2} f(t, 1).$$

Consequently, for any  $t \in [0, 1]$ , we have

$$\int_0^1 G(t, s)[f(s, [u(s) - x(s)]^*) + q_+(s)]ds$$

$$\begin{aligned}
&\leq \int_0^1 G(s, s) [f(s, [u(s) - x(s)]^*) + q_+(s)] ds \\
&\leq \int_0^1 G(s, s) [a^{(\lambda_2 - \lambda_1)} \|u\|^{\lambda_2} f(s, 1) + q_+(s)] ds \\
&\leq (a^{(\lambda_2 - \lambda_1)} \|u\|^{\lambda_2} + 1) \int_0^1 G(s, s) [f(s, 1) + q_+(s)] ds \\
&< +\infty.
\end{aligned}$$

If we define an operator  $T : P \rightarrow P$  by

$$Tu(t) = \int_0^1 G(t, s) [f(s, [u(s) - x(s)]^*) + q_+(s)] ds, \quad u \in P,$$

then we have the following lemma:

**Lemma 2.3.** Suppose that  $(H_1)$  and  $(H_2)$  hold. Then the operator  $T$  has a fixed point in  $C[0, 1]$  if and only if the following boundary value problem

$$\begin{cases} u'' + f(t, [u(t) - x(t)]^*) + q_+(t) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \quad (2.1)$$

has a  $C[0, 1] \cap C^2(0, 1)$  positive solution.

**Lemma 2.4.** Assume that  $(H_1)$  and  $(H_2)$  hold. Then  $T(Q) \subset Q$  and  $T : Q \rightarrow Q$  is a completely continuous operator.

**Proof.** For any  $u \in Q$ , let  $y(t) = Tu(t)$ . By definition of the operator  $T$ , we have  $y(0) = Tu(0) = 0$ ,  $y(1) = Tu(1) = 0$ . Hence there exists a  $t_0 \in (0, 1)$  such that  $\|y\| = y(t_0)$ . Since

$$\frac{G(t, s)}{G(t_0, s)} = \begin{cases} \frac{t}{t_0}, & t, t_0 \leq s, \\ \frac{t(1-s)}{s(1-t_0)}, & t \leq s \leq t_0, \\ \frac{1-t}{1-t_0}, & s \leq t, t_0, \\ \frac{s(1-t)}{t_0(1-s)}, & t_0 \leq s \leq t, \end{cases}$$

we obtain that

$$\frac{G(t, s)}{G(t_0, s)} \geq t(1-t), \quad (t, s) \in (0, 1) \times (0, 1).$$

Then

$$y(t) = \int_0^1 G(t, s) [f(s, [u(s) - x(s)]^*) + q_+(s)] ds$$

$$\begin{aligned}
&= \int_0^1 \frac{G(t, s)}{G(t_0, s)} G(t_0, s) [f(s, [u(s) - x(s)]^*) + q_+(s)] ds \\
&\geq t(1-t)y(t_0) = t(1-t)\|y\|, \quad t \in [0, 1].
\end{aligned}$$

Thus  $T(Q) \subset Q$ .

Let  $D \subset Q$  be any bounded set. Then there exists a constant  $L > 0$  such that  $\|u\| \leq L$  for any  $u \in D$ . Moreover, for any  $u \in D$ ,  $s \in [0, 1]$ , notice  $[u(s) - x(s)]^* \leq u(s) \leq \|u\| \leq L < L + 1$  and Lemma 2.2 and (1.5), then for any  $u \in D$ ,  $s \in [0, 1]$ , we have

$$\begin{aligned}
f(s, [u(s) - x(s)]^*) + q_+(s) &\leq f(s, L + 1) + q_+(s) \\
&\leq (L + 1)^{\lambda_1} f(s, 1) + q_+(s) \\
&\leq [(L + 1)^{\lambda_1} + 1][f(s, 1) + q_+(s)].
\end{aligned} \tag{2.2}$$

Consequently,

$$\begin{aligned}
|Tu(t)| &= \int_0^1 G(t, s) [f(s, [u(s) - x(s)]^*) + q_+(s)] ds \\
&\leq \int_0^1 G(s, s) [(L + 1)^{\lambda_1} + 1][f(s, 1) + q_+(s)] ds \\
&\leq [(L + 1)^{\lambda_1} + 1] \int_0^1 s(1-s)[f(s, 1) + q_+(s)] ds < +\infty.
\end{aligned}$$

Therefore  $T(D)$  is uniformly bounded.

Now we show that  $T(D)$  is equicontinuous on  $[0, 1]$ . For any  $u \in D$ ,  $t \in [0, 1]$ , we have

$$\begin{aligned}
\left| \frac{d}{dt} Tu(t) \right| &= \left| - \int_0^t s [f(s, [u(s) - x(s)]^*) + q_+(s)] ds \right. \\
&\quad \left. + \int_t^1 (1-s) [f(s, [u(s) - x(s)]^*) + q_+(s)] ds \right| \\
&\leq [(L + 1)^{\lambda_1} + 1] \left( \int_0^t s [f(s, 1) + q_+(s)] ds \right. \\
&\quad \left. + \int_t^1 (1-s) [f(s, 1) + q_+(s)] ds \right).
\end{aligned}$$

Combining with exchanging the integral sequence, we obtain

$$\begin{aligned}
& \int_0^1 \left( \int_0^t s[f(s, 1) + q_+(s)] ds + \int_t^1 (1-s)[f(s, 1) + q_+(s)] ds \right) dt \\
&= \int_0^1 ds \int_s^1 s[f(s, 1) + q_+(s)] dt + \int_0^1 ds \int_0^s (1-s)[f(s, 1) + q_+(s)] dt \\
&= 2 \int_0^1 s(1-s)[f(s, 1) + q_+(s)] ds < +\infty.
\end{aligned}$$

So, for any  $u \in D$ , we have

$$0 \leq \int_0^1 \left| \frac{d}{dt} Tu(t) \right| dt \leq 2[(L+1)^{\lambda_1} + 1] \int_0^1 s(1-s)[f(s, 1) + q_+(s)] ds < +\infty.$$

From the absolute continuity of the integral, we know  $T(Q)$  is equicontinuous on  $[0, 1]$ . Thus according to Ascoli–Arzela theorem,  $T(Q)$  is a relatively compact set.

In the end, we show  $T: Q \rightarrow Q$  is continuous. Assume  $u_n, u_0 \in Q$ ,  $u_n \rightarrow u_0$  ( $n \rightarrow +\infty$ ). Then there exists a constant  $L_1 > 0$  such that  $\|u_0\| \leq L_1$ ,  $\|u_n\| \leq L_1$  ( $n = 1, 2, \dots$ ). It is similar to (2.2) and we obtain

$$\begin{aligned}
f(s, [u_n(s) - x(s)]^*) + q_+(s) &\leq [(L_1 + 1)^{\lambda_1} + 1][f(s, 1) + q_+(s)], \\
n &= 1, 2, \dots
\end{aligned} \tag{2.3}$$

By definition of the operator  $T$ , we have

$$\begin{aligned}
|Tu_n(t) - Tu_0(t)| &\leq \int_0^1 G(s, s) |f(s, [u_n(s) - x(s)]^*) - f(s, [u_0(s) - x(s)]^*)| ds, \\
n &= 1, 2, \dots
\end{aligned} \tag{2.4}$$

Set

$$r_n(s) = G(s, s) |f(s, [u_n(s) - x(s)]^*) - f(s, [u_0(s) - x(s)]^*)|$$

and

$$F(s) = 2G(s, s)[(L_1 + 1)^{\lambda_1} + 1][f(s, 1) + q_+(s)], \quad s \in (0, 1).$$

From (2.3), we have

$$|r_n(s)| \leq F(s), \quad s \in (0, 1), \quad n = 1, 2, 3, \dots,$$

and  $\{r_n(s)\}$  is a measurable function sequence in  $(0, 1)$ . By  $(H_2)$ ,

$$0 \leq \int_0^1 F(s) ds = 2[(L_1 + 1)^{\lambda_1} + 1] \int_0^1 G(s, s)[f(s, 1) + q_+(s)] ds < +\infty.$$

We assert that  $r_n(s) \rightarrow 0$  ( $n \rightarrow +\infty$ ) for any fixed  $s \in (0, 1)$ . In fact, for any fixed  $s \in (0, 1)$ , in view of the continuity of  $f(s, u)$  in relative to  $u$ , for any  $\varepsilon > 0$ , there exists a constant  $\delta > 0$ , for any  $v_1, v_2 \geq 0$ , when  $|v_1 - v_2| < \delta$ , such that

$$|f(s, v_1) - f(s, v_2)| < \frac{\varepsilon}{G(s, s)}.$$

Since  $u_n(s) \rightarrow u_0(s)$ , then there exists a constant  $N > 0$  such that  $|u_n(s) - u_0(s)| < \delta$  for  $n > N$ . Notice that

$$\begin{aligned} & |[u_n(s) - x(s)]^* - [u_0(s) - x(s)]^*| \\ &= \left| \frac{|u_n(s) - x(s)| + u_n(s) - x(s)}{2} - \frac{|u_0(s) - x(s)| + u_0(s) - x(s)}{2} \right| \\ &= \left| \frac{|u_n(s) - x(s)| - |u_0(s) - x(s)|}{2} + \frac{u_n(s) - u_0(s)}{2} \right| \\ &\leq |u_n(s) - u_0(s)| < \delta. \end{aligned}$$

Then when  $n > N$ , we have

$$|f(s, [u_n(s) - x(s)]^*) - f(s, [u_0(s) - x(s)]^*)| < \frac{\varepsilon}{G(s, s)}.$$

Therefore, for any fixed  $s \in (0, 1)$ ,  $\forall \varepsilon > 0$ ,  $\exists N > 0$ , when  $n > N$ , we get

$$|r_n(s) - 0| = G(s, s) |f(s, [u_n(s) - x(s)]^*) - f(s, [u_0(s) - x(s)]^*)| < \varepsilon.$$

That is  $r_n(s) \rightarrow 0$  ( $n \rightarrow +\infty$ ),  $s \in (0, 1)$ .

Now, by using Lebesgue control convergence theorem, we have

$$\|Tu_n - Tu_0\| \leq \int_0^1 r_n(s) ds \rightarrow 0 \quad (n \rightarrow +\infty).$$

Therefore  $T: Q \rightarrow Q$  is continuous. Thus  $T: Q \rightarrow Q$  is a completely continuous operator.  $\square$

**Lemma 2.5.** Let  $Q_r = \{u \in Q: \|u\| < r\}$ . Then  $i(T, Q_r, Q) = 1$ .

**Proof.** Assume there exists  $\lambda_0 \geq 1$ ,  $z_0 \in \partial Q_r$  such that  $\lambda_0 z_0 = Tz_0$ . Then  $z_0 = \frac{1}{\lambda_0} Tz_0$  and  $0 < \frac{1}{\lambda_0} \leq 1$ . Since  $z_0(t) \geq \|z_0\|t(1-t) = rt(1-t)$ ,  $t \in [0, 1]$  and

$$x(t) = \int_0^1 G(t, s) q_-(s) ds \leq t(1-t) \int_0^1 q_-(s) ds.$$

This and  $(H_2)$  imply that, for any  $t \in [0, 1]$ ,

$$z_0(t) - x(t) \geq z_0(t) - t(1-t) \int_0^1 q_-(s) ds \geq rt(1-t) - t(1-t) \int_0^1 q_-(s) ds = 0.$$



Applying  $z_0 = \frac{1}{\lambda_0} T z_0$ , by direct computation, we obtain  $\lambda_0$  and  $z_0$  such that

$$\begin{cases} z_0''(t) + \frac{1}{\lambda_0} [f(s, z_0(s) - x(s)) + q_+(s)] = 0, \\ z_0(0) = z_0(1) = 0. \end{cases} \quad (2.5)$$

Since  $z_0''(t) \leq 0$  for any  $t$  in  $(0, 1)$ , then  $z_0(t)$  is a concave function on  $[0, 1]$ . This together with the boundary conditions imply that there exists  $t_0 \in (0, 1)$  such that

$$\begin{aligned} \|z_0\| &= z_0(t_0), & z_0'(t_0) &= 0, & z_0'(t) &\geq 0, & t \in (0, t_0), \\ z_0'(t) &\leq 0, & t \in (t_0, 1). \end{aligned}$$

Choose  $t \in (0, t_0)$ . Integrating (2.5) from  $t$  to  $t_0$ , we have

$$\begin{aligned} z_0'(t) &= \int_t^{t_0} -z_0''(s) ds \leq \int_t^{t_0} [f(s, z_0(s) - x(s)) + q_+(s)] ds \\ &\leq [(r+1)^{\lambda_1} + 1] \int_t^{t_0} [f(s, 1) + q_+(s)] ds. \end{aligned} \quad (2.6)$$

Then integrating (2.6) from 0 to  $t_0$ , we also have

$$\begin{aligned} r = z_0(t_0) &= \int_0^{t_0} z_0'(s) ds \\ &\leq [(r+1)^{\lambda_1} + 1] \int_0^{t_0} ds \int_s^{t_0} [f(\tau, 1) + q_+(\tau)] d\tau \\ &\leq [(r+1)^{\lambda_1} + 1] \int_0^{t_0} d\tau \int_0^\tau [f(\tau, 1) + q_+(\tau)] ds \\ &\leq [(r+1)^{\lambda_1} + 1] \int_0^{t_0} \tau [f(\tau, 1) + q_+(\tau)] d\tau \\ &\leq \frac{[(r+1)^{\lambda_1} + 1]}{1 - t_0} \int_0^1 \tau(1 - \tau) [f(\tau, 1) + q_+(\tau)] d\tau. \end{aligned}$$

Consequently,

$$\frac{r(1 - t_0)}{[(r+1)^{\lambda_1} + 1]} \leq \int_0^1 \tau(1 - \tau) [f(\tau, 1) + q_+(\tau)] d\tau. \quad (2.7)$$

In the same way, if  $t \in (t_0, 1)$ , we have

$$\frac{rt_0}{[(r+1)^{\lambda_1} + 1]} \leq \int_0^1 \tau(1-\tau)[f(\tau, 1) + q_+(\tau)] d\tau. \quad (2.8)$$

(2.7) and (2.8) imply that

$$\frac{r}{[(r+1)^{\lambda_1} + 1]} \leq 2 \int_0^1 \tau(1-\tau)[f(\tau, 1) + q_+(\tau)] d\tau,$$

which is a contradiction of (1.4). So applying Lemma 2.1,  $i(T, Q_r, Q) = 1$ .  $\square$

**Lemma 2.6.** *There exists a constant  $R > r$  such that  $i(T, Q_R, Q) = 0$ .*

**Proof.** Choose constants  $\alpha, \beta$  and  $M$  such that

$$[\alpha, \beta] \subset (0, 1), \quad M > 2 \left[ \alpha(1-\beta) \left( \max_{0 \leq t \leq 1} \int_{\alpha}^{\beta} G(t, s) ds \right) \right]^{-1}.$$

From Lemma 2.2, there exists  $R_1 > 2r$ , when  $t \in [\alpha, \beta]$ ,  $u \geq R_1$  such that

$$\frac{f(t, u)}{u} \geq \min_{t \in [\alpha, \beta]} \frac{f(t, u)}{u} \geq M.$$

That is

$$f(t, u) \geq Mu, \quad t \in [\alpha, \beta], \quad u \geq R_1.$$

Let  $R \geq \frac{2R_1}{\alpha(1-\beta)}$ . Obviously,  $R > R_1 > 2r$ . Thus  $r/R < 1/2$ . Now we show that  $u \not\geq Tu$ ,  $u \in \partial Q_R$ . In fact, otherwise, there exists  $y_1 \in \partial Q_R$  such that  $y_1 \geq Ty_1$ . As the proof of Lemma 2.5, for any  $t \in [\alpha, \beta]$ , we have

$$\begin{aligned} y_1(t) - x(t) &\geq y_1(t) - t(1-t) \int_0^1 q_-(s) ds \\ &= y_1(t) - t(1-t)r \geq y_1(t) - \frac{y_1(t)}{\|y_1\|} r \\ &= y_1(t) - \frac{r}{R} y_1(t) \geq \frac{1}{2} y_1(t) \geq \frac{1}{2} t(1-t) \|y_1\| \\ &\geq \frac{1}{2} R \alpha (1-\beta) \geq R_1 > 0. \end{aligned}$$

So

$$\begin{aligned} R \geq y_1(t) &\geq Ty_1(t) = \int_0^1 G(t, s) [f(s, [y_1(s) - x(s)]^*) + q_+(s)] ds \\ &= \int_0^1 G(t, s) [f(s, y_1(s) - x(s)) + q_+(s)] ds \end{aligned}$$

$$\begin{aligned}
&\geq \int_{\alpha}^{\beta} G(t, s) [f(s, y_1(s) - x(s)) + q_+(s)] ds \\
&\geq \int_{\alpha}^{\beta} G(t, s) f(s, y_1(s) - x(s)) ds \\
&\geq \int_{\alpha}^{\beta} G(t, s) M [y_1(s) - x(s)] ds \\
&\geq \int_{\alpha}^{\beta} G(t, s) \frac{1}{2} R \alpha (1 - \beta) M ds \\
&\geq \frac{1}{2} M \alpha (1 - \beta) R \int_{\alpha}^{\beta} G(t, s) ds, \quad t \in [0, 1].
\end{aligned}$$

Consequently,

$$R \geq \frac{1}{2} M \alpha (1 - \beta) R \max_{0 \leq t \leq 1} \int_{\alpha}^{\beta} G(t, s) ds.$$

That is

$$M \leq 2 \left[ \alpha (1 - \beta) \left( \max_{0 \leq t \leq 1} \int_{\alpha}^{\beta} G(t, s) ds \right) \right]^{-1}.$$

This contradicts  $M$  that we choose. Thus from Lemma 2.1,  $i(T, Q_R, Q) = 0$ .  $\square$

### 3. Proof of main results

**Proof of Theorem 1.1.** Applying Lemmas 2.5 and 2.6 and the definition of the fixed point index, we have  $i(T, Q_R \setminus \overline{Q_r}, Q) = -1$ . Thus  $T$  has a fixed point  $z_0(t)$  in  $Q_R \setminus \overline{Q_r}$  with  $r < \|z_0\| < R$ . Since  $\|z_0\| > r$ , we have

$$\begin{aligned}
z_0(t) - x(t) &\geq \|z_0\| t(1 - t) - \int_0^1 G(t, s) q_-(s) ds \\
&= \|z_0\| t(1 - t) - t(1 - t) \int_0^1 q_-(s) ds \\
&= (\|z_0\| - r) t(1 - t) = kt(1 - t) \geq 0, \quad t \in [0, 1].
\end{aligned}$$

It follows from Lemma 2.3 that  $z_0(t)$  satisfies

$$\begin{cases} z_0''(t) + f(t, z_0(t) - x(t)) + q_+(t) = 0, & 0 < t < 1, \\ z_0(0) = z_0(1) = 0. \end{cases}$$

Let  $u_0(t) = z_0(t) - x(t)$ ,  $t \in [0, 1]$ . Noticing  $u_0''(t) = z_0''(t) + q_-(t)$ ,  $t \in (0, 1)$ , so we have

$$\begin{cases} u_0''(t) + f(t, u_0(t)) + q(t) = 0, & 0 < t < 1, \\ u_0(0) = u_0(1) = 0. \end{cases}$$

Therefore  $u_0(t)$  is a  $C[0, 1] \cap C^2(0, 1)$  positive solution of BVP (1.2), and exists a constant  $k > 0$  such that  $u_0(t) \geq kt(1-t)$ ,  $t \in [0, 1]$ . The proof of Theorem 1.1 is completed.  $\square$

Finally we give an example as an application of Theorem 1.1. Choose

$$f(t, u) = \frac{u^{3/2}}{20t(1-t)}, \quad q_-(t) = \frac{1}{\sqrt{t}}, \quad q_+(t) \equiv 0, \quad r = \int_0^1 q_-(t) dt = 2,$$

and  $\lambda_1 = 2 > \lambda_2 = 5/4 > 1$ , then  $(H_1)$  is satisfied. Notice  $(H_2)$  also holds, since

$$2 \int_0^1 s(1-s)[f(s, 1) + q_+(s)] ds = \frac{1}{10} < \frac{1}{5}.$$

Existence of a  $C[0, 1] \cap C^2(0, 1)$  positive solution is now guaranteed from Theorem 1.1.

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